

# A PATH ALGORITHM FOR AFFINE KAZHDAN-LUSZTIG POLYNOMIALS

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Let  $\lambda, \mu$  be dominant integral weights of a semisimple Lie algebra  $\mathfrak{g}$  and let  $l$  be a positive integer larger than the dual Coxeter number of  $\mathfrak{g}$ . Using these data and certain parabolic Kazhdan-Lusztig polynomials for affine Weyl groups, one can define polynomials  $n_{\lambda, \mu}$  in  $\mathbb{Z}[v]$ . It has been shown recently that these polynomials can be used to compute the character of indecomposable tilting modules of quantum groups at a root of unity (see [S1,2]). For Lie type  $A$ , these polynomials also describe the structure of the finite Hecke algebras of type  $A$  for  $q$  a primitive  $l$ -th root of unity. This was derived by a different method (see [A], [G], [LLT]), motivated by Kashiwara's crystal bases; the coincidence of these polynomials was shown in [GW2] and [VV]. The main result of our paper is an algorithm for computing the polynomials  $n_{\lambda, \mu}$  using piecewise linear paths, for all Lie types. This was, in part, inspired by the algorithm of Lascoux, Leclerc, and Thibon [LLT], which was used in the Hecke algebra setting. So our algorithm can be viewed as a generalization of the LLT algorithm to arbitrary weight lattices. However, the method of proof here relies heavily on Kazhdan-Lusztig theory.

One of the motivations for this work was to study in more detail the behaviour of the polynomials  $n_{\lambda, \mu}$  with  $\lambda, \mu$  near the boundary of the Weyl chamber. As already explained in our previous paper [GW2] for type  $A$ , the algorithm is considerably faster than previous algorithms; this allows a more extensive empirical investigation of the polynomials  $n_{\lambda, \mu}$ . Moreover, our algorithm shows some similarities with Littelmann's path algorithm which describes the decomposition of tensor products of Kac-Moody algebras. Indeed, at least in some special cases, our algorithm can be used to compute the decomposition of the tensor product of tilting modules into a direct sum of indecomposable tilting modules. This could be used to compute the dimensions of simple modules of the finite Hecke algebras of type  $A$ , using the  $q$ -analogue of Schur-Weyl duality. Moreover, for Lie type  $A$ , the path operators can be identified with operators appearing in the Fock space representation of  $U_q \widehat{sl_l}$ . It would be interesting if one could find a similar interpretation of these operators also for other Lie types. This, as well as the connection to tilting modules and Hecke algebras are discussed at the end of section 3.

Our paper is organized as follows. In section 1 we first review basic facts of affine Hecke algebras and their Kazhdan-Lusztig polynomials. We

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then study the multiplication of parabolic Kazhdan-Lusztig elements by Kazhdan-Lusztig elements  $C_P$  corresponding to the longest element of an arbitrary parabolic subgroup  $P$ . In Section 2, we define a geometric procedure which roughly corresponds to such multiplications. In Section 3 we describe the already mentioned algorithm and discuss various other (potential) applications.

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## 1. AFFINE HECKE ALGEBRAS

We review basic facts of Kazhdan Lusztig polynomials for affine Hecke algebras, following Soergel [S1].

1.1. Let  $\Delta$  be a root system of a simple Lie algebra,  $Y$  its root lattice and  $X$  its weight lattice. Moreover, let  $V = X \otimes_{\mathbb{Z}} \mathbb{R}$ . The highest root is denoted by  $\theta$ . We fix a bilinear form  $\cdot$  on the  $\mathbb{Z}$  span of  $\Delta$  uniquely determined by  $\alpha \cdot \alpha = 2$  for all short roots  $\alpha$  (see [Lu, 1.1]). We obtain an embedding of the root lattice into the weight lattice via this bilinear form.

Define for each positive root  $\alpha$  the hyperplane  $E_{\alpha} = \{x \in V, x \cdot \alpha = 0\}$ . More generally, we define for any integer  $r$  and any positive root  $\alpha$  the affine hyperplane

$$E_{\alpha,r} = \{x \in V, x \cdot \alpha = r\}.$$

Let  $\mathcal{W}$  be the affine reflection group generated by the finite Weyl group  $W$  corresponding to  $\Delta$  and the affine reflection in the hyperplane  $E_0 = E_{\theta,l}$  where  $\theta$  is the highest root in  $\Delta$ . The image of the defining action of an element  $w \in \mathcal{W}$  on  $x \in V$  is denoted by  $w(x)$ . Let  $\rho$  be half the sum of all positive roots in  $\Delta$ . Then we define the dot action of  $\mathcal{W}$  by the formula  $w.x = w(x + \rho) - \rho$ .

It can be shown that the collection  $\mathcal{E} = \{E_{\alpha,ml}, \alpha \in \Delta_+, m \in \mathbb{Z}\}$  partitions  $V$  into a disjoint collection of fundamental domains, called alcoves, with respect to  $\mathcal{W}$ . The hyperplane  $E_{\alpha} = E_{\alpha,0}$  divides  $V$  into two (open) halfspaces  $E_{\alpha}^+ = \{x \in V, (x, \alpha) > 0\}$  and  $E_{\alpha}^-$ , the other half space. The half spaces  $E_{\alpha,m}^+$  and  $E_{\alpha,m}^-$  are the images of  $E_{\alpha}^+$  resp  $E_{\alpha}^-$  under a translation which maps  $E_{\alpha}$  to  $E_{\alpha,m}$ . The dominant Weyl chamber  $\mathcal{C}$  is equal to the intersection of  $E_{\alpha}^+$ , with  $\alpha$  a simple root. We say that a hyperplane  $E \in \mathcal{E}$  contains a lower wall of the alcove  $A$  if  $A$  is on the positive side of  $E$ , i.e.  $A \subset E^+$  and the intersection of  $E$  with the closure of  $A$  has codimension 1; similarly, we say that  $E$  contains an upper wall of  $A$  if  $A \subset E^-$  and the intersection of  $E$  with the closure of  $A$  has codimension 1.

The fundamental alcove  $A_+$  is defined to be the intersection of  $\mathcal{C}$  with  $E_{\theta,l}^-$ , where  $\theta$  is the highest root. The fundamental box  $\Pi$  is the set of points  $x \in \mathcal{C}$  which satisfy  $0 < x \cdot \alpha \leq l$  for all simple roots  $\alpha$ . The set of alcoves in the dominant Weyl chamber is denoted by  $\mathcal{A}_+$ .

A facet  $F$  is a nonempty connected subset of  $V$  such that for any root  $\alpha$  either  $F$  is between two neighboring hyperplanes in  $\mathcal{E}$  which are parallel to  $E_\alpha$  (with empty intersection with both of those hyperplanes), or it is contained in  $E_{\alpha, ml}$  for some  $m \in \mathbb{Z}$ . For a facet  $F$  we define  $A_+(F)$  to be the alcove which contains  $F$  in its boundary and which is on the positive side of each hyperplane which contains  $F$ . The alcove  $A_-(F)$  is defined similarly.

**1.2. Affine Hecke algebras.** The affine Hecke algebra  $\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S})$  is the associative algebra with identity element  $\mathbf{1}$  over the ring of Laurent polynomials  $\mathbb{Z}[v, v^{-1}]$  with generators  $\{T_s : s \in \mathcal{S}\}$  satisfying the braid relations and the quadratic relation  $T_s^2 = v^{-2}\mathbf{1} + (v^{-2} - 1)T_s$ , for  $s \in \mathcal{S}$ . Using the generators  $H_s = vT_s$ , one has instead the quadratic relations  $H_s^2 = \mathbf{1} + (v^{-1} - v)H_s$ . The Hecke algebra has a basis  $\{H_x : x \in \mathcal{W}\}$  satisfying  $H_x H_y = H_{xy}$  in case  $\ell(xy) = \ell(x) + \ell(y)$ , and  $H_s H_x = H_{sx} + (v^{-1} - v)H_x$  in case  $\ell(sx) = \ell(x) - 1$ , for  $x \in \mathcal{W}$  and  $s \in \mathcal{S}$ ; here  $\ell$  denotes the usual length function on a Coxeter group with respect to the generating set  $\mathcal{S}$ . The Hecke algebra has an involution  $d : a \mapsto \bar{a}$  defined by  $\bar{v} = v^{-1}$  and  $(H_x)^- = (H_{x^{-1}})^{-1}$ . An element fixed by this involution is called self-dual. We denote the self-dual part of  $\mathbb{Z}[v, v^{-1}]$  by  $\mathbb{Z}[v, v^{-1}]_0$ . Fundamental self-dual elements are the  $C_s = H_s + v$  for  $s \in \mathcal{S}$ . Let  $P$  be the parabolic subgroup generated by a subset  $\mathcal{S}_P$  of  $\mathcal{S}$ . We will assume here that  $\mathcal{S}_P$  is a proper subset of  $\mathcal{S}$ , which implies that  $P$  is finite. We shall denote by  $C_P$  the Kazhdan-Lusztig element in the Hecke algebra  $\mathcal{H}(P, \mathcal{S}_P)$  corresponding to the longest element  $w_0$  of  $P$ . It is well-known that in our notations  $C_P$  is given by the formula

$$(1.1) \quad C_P = \sum_{w \in P} v^{\ell_P - \ell(w)} H_w,$$

where  $\ell_P = \ell(w_0)$ . It is also well-known and easy to derive that for any  $w \in P$  we have  $H_w C_P = v^{-\ell(w)} C_P$ . The  $v$ -order  $[P]_v$  of  $P$  is defined by

$$[P]_v = v^{-\ell_P} \sum_{w \in P} v^{2\ell(w)} = v^{\ell_P} \sum_{w \in P} v^{-2\ell(w)}.$$

If  $Q \subset P$  is a parabolic subgroup of  $P$ , and the element  $C_Q$  is defined accordingly, we can easily derive from the formulas mentioned before that

$$(1.2) \quad C_P C_Q = [Q]_v C_P = C_Q C_P.$$

**1.3. Parabolic module.** Let  $\mathcal{S}_0 \subseteq \mathcal{S}$  be the set of simple reflections fixing the origin. The finite Weyl group  $W$  is the Coxeter subgroup of  $\mathcal{W}$  with generating set  $\mathcal{S}_0$ . Let  $\mathcal{H}_f = \mathcal{H}(W, \mathcal{S}_0) \subseteq \mathcal{H}$  denote its Hecke algebra. Consider the sign representation of  $\mathcal{H}_f \rightarrow \mathbb{Z}[v, v^{-1}]$  which takes each  $H_s$  to  $-v^{-1}$ . Let  $\mathcal{N}$  denote the induced right  $\mathcal{H}$ -module

$$\mathcal{N} = \mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}_f} \mathcal{H}.$$

Let  $\mathcal{W}^f \subseteq \mathcal{W}$  be the coset representatives of minimal length of the right cosets of  $W$  in  $\mathcal{W}$ . Then  $\mathcal{N}$  has a basis  $N_x = \mathbf{1} \otimes H_x$ , for  $x \in \mathcal{W}^f$ , and the operation of the  $C_s$  for  $s \in \mathcal{S}$  on this basis has the following form:

$$(1.3) \quad N_x C_s = \begin{cases} N_{xs} + v N_x & \text{if } xs \in \mathcal{W}^f \text{ and } xs > x \\ N_{xs} + v^{-1} N_x & \text{if } xs \in \mathcal{W}^f \text{ and } xs < x \\ 0 & \text{if } xs \notin \mathcal{W}^f, \end{cases}$$

where the inequality signs refer to the Bruhat order on  $\mathcal{W}$ . The involution on  $\mathcal{H}$  induces an involution on  $\mathcal{N}$  defined by  $a \otimes b \mapsto \bar{a} \otimes \bar{b}$ .

The following result is discussed in [S1], following [D1], [D2].

**Theorem 1.1.**  $\mathcal{N}$  has a unique basis  $\{\underline{N}_x : x \in \mathcal{W}^f\}$  satisfying

1.  $\underline{N}_x \in N_x + \sum_{y < x} v\mathbb{Z}[v]N_y$ .
2.  $\underline{N}$  is self-dual.

**Definition 1.2.** The affine Kazhdan-Lusztig polynomials  $n_{y,x}$  are defined by

$$\underline{N}_x = N_x + \sum_{y < x} n_{y,x} N_y$$

The affine Weyl group  $\mathcal{W}$  acts freely and transitively on alcoves, so there is a bijection  $\mathcal{W} \rightarrow \mathcal{A}$  given by  $w \mapsto wA_+$ , where  $A_+$  is the unique alcove in  $\mathcal{A}_+$  containing 0 in its closure. Under this bijection, the elements of  $\mathcal{W}^f$  correspond to alcoves contained in the positive Weyl chamber. One also has an action of  $\mathcal{W}$  on the right on  $\mathcal{A}$  given by  $(wA_+)x = wxA_+$ .

Using the bijection between  $\mathcal{W}^f$  and  $\mathcal{A}_+$ , one can rename the distinguished elements of the right  $\mathcal{H}$  module  $\mathcal{N}$  using alcoves  $A \in \mathcal{A}_+$  rather than coset representatives  $x \in \mathcal{W}^f$ . Thus if  $x, y \in \mathcal{W}^f$  correspond to  $A, B \in \mathcal{A}_+$ , then we write  $N_A$  for  $N_x$ ,  $\underline{N}_A$  for  $\underline{N}_x$ , and  $n_{A,B}$  for  $n_{x,y}$ . The right action of  $\mathcal{H}$  is then given by

$$(1.4) \quad N_A C_s = \begin{cases} N_{As} + v N_A & \text{if } As \in \mathcal{A}_+ \text{ and } As \succ A \\ N_{As} + v^{-1} N_A & \text{if } As \in \mathcal{A}_+ \text{ and } As \prec A \\ 0 & \text{if } As \notin \mathcal{A}_+, \end{cases}$$

where now the inequalities have a geometric interpretation:  $As \succ A$  if  $As$  is on the positive side of the hyperplane separating the two alcoves. The elements  $\underline{N}_A$  are computed by a recursive scheme as follows: One has  $\underline{N}_{A_+} = N_{A_+}$ . Given  $A \neq A_+$ , one can choose  $s \in \mathcal{S}$  such that  $As \in \mathcal{A}_+$  and  $As \prec A$ . The element  $\underline{N}_A$  is then computed by

$$\underline{N}_A = \underline{N}_{As} C_s - \sum_{B \prec A} f_{B,A}(0) \underline{N}_B,$$

where  $f_{B,A}$  is the coefficient of  $B$  in  $\underline{N}_{As} C_s$ .

1.4. Let  $F$  be a facet. The left-stabilizer  $\tilde{P}$  of  $F$  is generated by the reflections in the hyperplanes containing  $F$ . Let  $\mathcal{A}_F$  be the set of all alcoves  $A$  such that  $F$  is in the boundary of  $A$ . Then it is well-known (see e.g. [Ja, 6.11]) that there exists a 1-1 correspondence between the elements of  $\tilde{P}$  and the elements of the set  $\mathcal{A}_F$ ; it can be defined by the map  $w \in \tilde{P} \mapsto w(A_+(F))$ . In particular,  $\tilde{P}$  is a finite reflection group, which is generated by a subset of reflections in the hyperplanes containing a lower wall of  $A_+(F)$ .

Let  $w_F \in \mathcal{W}$  be the element such that  $A_+(F) = w_F(A_+)$ . Then we define the right-stabilizer of  $F$  to be the subgroup  $P = w_F^{-1}\tilde{P}w_F$ ; one checks easily that it does indeed coincide with all the elements  $w \in \mathcal{W}$  such that  $\mathcal{A}_F w = \mathcal{A}_F$ , and hence  $Fw = F$ . It is generated by a subset  $\mathcal{S}_F = \mathcal{S}_P \subset \mathcal{S}$  of simple reflections. Then it follows from the definition of right action and the discussion above that the map

$$(1.5) \quad w \in P \mapsto A_-(F)w$$

defines a 1-1 correspondence between the elements of  $P$  and the elements of the set  $\mathcal{A}_F$ . Moreover, if all the alcoves of  $\mathcal{A}_F$  are contained in  $\mathcal{C}$ , this correspondence is order preserving between the Bruhat order of  $P$  and the order  $\prec$  on  $\mathcal{A}_F$ .

Let  $F_0$  be a facet which contains  $F$  in its closure. We define the quantities  $\mathbf{a}_F(F_0)$  and  $\mathbf{b}_F(F_0)$  to be the number of hyperplanes containing  $F$  which are ‘above’ respectively ‘below’  $F_0$ ; a hyperplane  $H$  is said to be above  $F_0$  if  $F_0$  is on the negative side of  $H$ , with a similar definition for ‘below’. We get the following formulas which immediately follow from the definitions:

$$(1.6) \quad \mathbf{b}_F(F_0) = \mathbf{b}_F(A_-(F_0)) \quad \text{and} \quad \mathbf{a}_F(F_0) = \mathbf{a}_F(A_+(F_0)).$$

Also observe that if  $w \in P$  and  $A = A_-(F)w$ , then

$$(1.7) \quad \mathbf{b}_F(A) = \ell(w) \quad \text{and} \quad \mathbf{a}_F(A) = \ell_P - \ell(w).$$

1.5. We now consider the action of the Hecke algebra  $\mathcal{H}(\mathcal{W}, \mathcal{S})$  on the module  $\mathcal{N}$ . Let  $\mathcal{H}(P, \mathcal{S}_P)$  be the Hecke algebra corresponding to the right stabilizer  $P$  of the facet  $F$ . If all the alcoves of  $\mathcal{A}_F$  are in  $\mathcal{C}$ , it follows from equation 1.5 and the text below it that the map

$$(1.8) \quad H_w \in \mathcal{H}(P, \mathcal{S}_P) \mapsto N_{A_-(F)w} \in \mathcal{A}_F$$

defines an isomorphism of  $\mathcal{H}(P, \mathcal{S}_P)$  right modules. The definition of the Kazhdan-Lusztig element  $C_P$ , see 1.1, also suggests defining the element

$$(1.9) \quad N_F = \sum_{A \in \mathcal{A}_F} v^{\mathbf{a}_F(A)} N_A = N_{A_-(F)} C_P,$$

where the last equality follows from 1.7 and 1.8. For a given facet  $F$ , we define a partial order  $\prec$  on the set  $\{N_{w(F)}, w \in \mathcal{W} \text{ such that } w(F) \subset \mathcal{C}\}$  by

$$(1.10) \quad F_1 \prec F_2 \quad \leftrightarrow \quad A_+(F_1) \prec A_+(F_2).$$

Finally, let  $F_0$  be a facet which contains  $F$  in its boundary, and let  $Q \subset P$  be its stabilizer. Recall that  $[Q]_v = v^{-\ell_Q} \sum_{w \in Q} v^{2\ell(w)}$ . It follows from the definitions of  $N_F$  and  $N_{F_0}$  that

$$(1.11) \quad N_F = \sum_{F'_0} v^{\mathbf{a}_F(F'_0)} N_{F'_0},$$

where the summation goes over the facets of the form  $F_0 w$  with  $w \in P$ . The following elementary lemma is probably well-known to experts in Kazhdan-Lusztig theory.

**Lemma 1.3.** *Let  $F_0$ ,  $F$ ,  $Q$  and  $P$  be as just defined, with  $F$  in the closure of the dominant Weyl chamber  $\mathcal{C}$ , and let  $A \in \mathcal{A}_F$ . Then we have*

$$(1.12) \quad N_A C_P = \begin{cases} 0 & \text{if } F \not\subset \mathcal{C}, \\ v^{-\mathbf{b}_F(A)} N_F & \text{otherwise,} \end{cases}$$

and

$$(1.13) \quad N_{F_0} C_P = \begin{cases} 0 & \text{if } F \not\subset \mathcal{C}, \\ [Q]_v v^{-\mathbf{b}_F(F_0)} N_F & \text{otherwise,} \end{cases}$$

*Proof.* Let  $F$  be a facet in the boundary of  $\mathcal{C}$  such that  $F \not\subset \mathcal{C}$ , and let  $A$  be an alcove in  $\mathcal{A}_F$ . We claim that  $N_A C_P = 0$ . There exists an alcove  $B \in \mathcal{A}_F$ ,  $B \subset \mathcal{C}$  which has a wall in a boundary hyperplane of  $\mathcal{C}$ . Let  $s$  be the simple reflection corresponding to this wall. Then  $N_B C_s = 0$  (see e.g. [S1], or Section 1.3); but then also  $(v + v^{-1})N_B C_P = N_B C_s C_P = 0$ , which shows the claim for  $A = B$ . The claim will follow inductively if we can prove it for any alcove  $A \in \mathcal{A}_F$  which shares a wall with an alcove  $B \in \mathcal{A}_F$  for which it has already been established. Let  $s$  be the simple reflection corresponding to such a wall, and observe that  $s \in P$ . Then  $N_A = N_{Bs} = N_B C_s - v^{\pm 1} N_B$ , from which one easily deduces the claim for  $A$ .

Assume now that  $F \subset \mathcal{C}$ . Then necessarily any alcove  $A \in \mathcal{A}_F$  has to be in  $\mathcal{C}$ . If  $A = A_-(F)$ , we get  $N_{A_-(F)} C_P = N_F$  from 1.8 and the definitions of  $N_F$  and of  $C_P$ , see 1.9 and 1.1. If  $A = A_-(F)w$  for some  $w \in P$ , then

$$N_A C_P = N_{A_-(F)} H_w C_P = v^{-\ell(w)} N_{A_-(F)} C_P = v^{-\mathbf{b}_F(A)} N_F,$$

using 1.7 and 1.8. To prove the second claim, observe that because of  $Q \subset P$  we get

$$N_{F_0} C_P = N_{A_-(F_0)} C_Q C_P = [Q]_v N_{A_-(F_0)} C_P = [Q]_v v^{-\mathbf{b}_F(F_0)} N_F,$$

where the first equality follows from the definition of  $N_{F_0}$  (formula 1.9), the second equality from formula 1.2, and the last equality from the formulas 1.12 for  $N_A C_P$  and 1.6 for  $\mathbf{b}_F(F_0)$ .  $\square$

1.6. We define the self-dual element  $\underline{N}_F$  by  $\underline{N}_F = \underline{N}_{A_+(F)}$ . Observe that for two facets  $F, \tilde{F}$ , not necessarily of same dimension we have  $\underline{N}_F = \underline{N}_{\tilde{F}}$  iff  $A_+(F) = A_+(\tilde{F})$ .

**Lemma 1.4.** *Let  $F$  be a facet with right stabilizer  $P$ , and let  $F_0$  be a facet with right stabilizer  $Q \subseteq P$ , such that  $F_0$  contains the facet  $F$  in its boundary. Write  $\underline{N}_{F_0} = \sum_D r_D N_D$ , with  $D \in \mathcal{A}_+$ . Then*

- (a) *For any  $A \in \mathcal{A}_F$ ,  $\underline{N}_A C_P$  is a  $\mathbb{Z}[v, v^{-1}]_0$  linear combination of elements of the form  $\underline{N}_B$  with  $B \prec A_+(F)$  (or equal),*
- (b)  $\underline{N}_{F_0} = \sum_{F'_0 \in \mathcal{W}(F_0)} r_{F'_0} N_{F'_0}$ , with  $r_{F'_0} = r_{A_+(F'_0)}$ ,
- (c)  $\underline{N}_{F_0} C_P = [Q]_v \sum_{F'_0 \in \mathcal{W}(F_0)} r_{F'_0} v^{-\mathbf{b}_{F'}(F'_0)} N_{F'}$ , where  $F'$  denotes, for each  $F'_0 \in \mathcal{W}(F_0)$ , the unique facet conjugate to  $F$  which is contained in the closure of  $F'_0$ .
- (d) *Let  $F_1$  be any facet such that  $F_1$  contains  $F$  in its boundary. Then*

$$\underline{N}_{F_0} C_P = [Q]_v \sum_{F'_0 \in \mathcal{W}(F_0)} \sum_{F'_1} r_{F'_0} v^{\mathbf{a}_{F'}(F'_1) - \mathbf{b}_{F'}(F'_0)} N_{F'_1},$$

where  $F'$  is related to  $F_0$  as in part (c), and  $F'_1$  ranges over all facets in  $\mathcal{W}(F_1)$  which contain  $F'$  in their closure.

*Proof.* To prove part (a), observe that  $C_P$  is equal to the Kazhdan-Lusztig element  $C_{w_0}$  of the longest element  $w_0$  in  $P$ . In particular, it is self-dual and hence so is  $\underline{N}_A C_P$ . Hence this product can be written as a  $\mathbb{Z}[v, v^{-1}]_0$  linear combination of self-dual elements  $\underline{N}_B$ ; in fact the same statement is true for  $\underline{N}_A C_w$  for any  $w \in P$ . We will show the second statement of the claim more generally for  $\underline{N}_A C_w$  for any  $w \in P$  by induction on the length  $\ell(w)$  of  $w$ . If  $\ell(w) = 1$ ,  $w = s$  is a simple reflection of  $P$ . In this case the claim follows from the original proof by Kazhdan and Lusztig: in fact,  $\underline{N}_A C_P$  is a linear combination of  $\underline{N}_B$  with  $B$  majorized by the larger of  $A$  or  $As$ . Either of those elements contains  $F$  in its boundary, and hence is majorized by  $A_+(F)$ . For the induction step, let  $w = w's$ , with  $\ell(w') < \ell(w)$  and  $s \in \mathcal{S}_P$ . Then the claim follows by applying the induction assumption twice for  $\underline{N}_A C_w = \underline{N}_A C_{w'} C_s$ .

Now let  $A$  denote  $A_+(F_0)$ . Let  $\mathcal{S}_Q$  be the set of simple reflections  $s \in Q$ . By definition of  $A = A_+(F_0)$  and  $Q$ , we have  $As \prec A$  for  $s \in \mathcal{S}_Q$ . But then  $\underline{N}_A C_s = (v + v^{-1})\underline{N}_A$  and  $\underline{N}_A H_s = v^{-1}\underline{N}_A$  for all  $s \in \mathcal{S}_Q$ , using basic properties of Kazhdan-Lusztig elements (see [KL]). But then also  $\underline{N}_A H_w = v^{-\ell(w)}\underline{N}_A$  for all  $w \in Q$  from which one easily concludes  $\underline{N}_A C_Q = [Q]_v \underline{N}_A$ .

Let  $D$  be an alcove, and let  $F'_0$  be the facet in the boundary of  $D$  which is conjugate to  $F_0$ . Then it follows from Lemma 1.3 that  $N_D C_Q$  is equal to a multiple of  $N_{F'_0}$ , and hence also  $\underline{N}_{F'_0} C_Q$  is a linear combination of  $N_{F'_0}$ 's. On the other hand, we have  $\underline{N}_{F'_0} C_Q = [Q]_v \underline{N}_{F'_0}$  by the previous paragraph of this proof. Hence already  $\underline{N}_{F'_0}$  itself is a linear combination of  $N_{F'_0}$ 's. From

this follows claim (b), as the coefficient of  $N_{F'_0}$  is equal to  $r_{A_+(F'_0)}$ . Using (b) and Lemma 1.3,(b), we get

$$\underline{N}_{F_0} C_P = \sum_{F'_0} r_{F'_0} N_{F'_0} C_P = [Q]_v \sum_{F'_0} r_{F'_0} v^{-\mathbf{b}_{F'_0}(F'_0)} N_{F'_0},$$

where  $F'$  is the unique facet conjugate to  $F$  which is contained in the closure of  $F'_0$ . This proves part (c), and part (d) follows from  $N_{F'} = \sum_{F'_1} v^{\mathbf{a}_{F'}(F'_1)} N_{F'_1}$  (see 1.11).  $\square$

In our application of part (d) of this lemma,  $F_1$  will be related to  $F_0$  as follows:  $F_1$  will be the facet “on the other side of  $F$  from  $F_0$ ,” to be more precise, it is the facet reached if one extends a line connecting a point in  $F_0$  with a point in  $F$  a little beyond  $F$ . Observe that  $F_1$  then lies in the affine subspace spanned by  $F_0$  and has the same dimension as  $F_0$ .

## 2. PATH OPERATORS

2.1. We now define Kazhdan-Lusztig polynomials  $n_{\lambda\mu}$  for *arbitrary points*  $\lambda, \mu$  in the dominant Weyl chamber  $\mathcal{C}$  of  $\mathfrak{g}$  as follows: Let  $\mu$  be a point in  $\mathcal{C}$ , and let  $F$  be the facet containing  $\mu + \rho$ . If  $\mu + \rho$  does not lie on any facet, we define both  $F$  and  $A_+(\mu)$  to be the alcove containing  $\mu + \rho$ ; otherwise, we define  $A_+(\mu)$  to be equal to  $A_+(F)$ . Then the polynomials  $n_{\lambda\mu}$  are defined by

$$(2.1) \quad n_{\lambda\mu} = \begin{cases} n_{A_+(\lambda), A_+(\mu)} & \text{if } \lambda + \rho \in \mathcal{W}(\mu + \rho), \\ 0 & \text{otherwise.} \end{cases}$$

The elements  $\underline{N}_\mu$  are defined in the free  $\mathbb{Z}[v, v^{-1}]$  module  $\mathcal{M}$  with basis  $\{N_\lambda, \lambda \in \mathcal{C}\}$  by

$$(2.2) \quad \underline{N}_\mu = \sum_{\lambda} n_{\lambda\mu} N_\lambda.$$

By definition, the polynomials  $n_{w, \mu, \mu}$  do not change if we vary  $\mu$  within its facet, and keep  $w \in \mathcal{W}$  fixed. If we denote by  $F(\lambda + \rho)$  the facet containing  $\lambda + \rho$ , and if  $\mu + \rho$  is on the facet  $F_0$ , we get

$$(2.3) \quad \underline{N}_{F_0} = \underline{N}_{A_+(F_0)} = \sum_{\lambda} n_{\lambda\mu} N_{F(\lambda+\rho)}.$$

An element  $M$  in  $\mathcal{M}$  is said to be self-dual if it can be rewritten as a linear combination of elements of the form  $\underline{N}_\mu$  with coefficients in  $\mathbb{Z}[v, v^{-1}]_0$ . If  $M = \sum_{\lambda} m_{\lambda} N_{\lambda}$ , it follows that  $M$  is self-dual if and only if the element  $\sum_{\lambda} m_{\lambda} N_{F(\lambda+\rho)}$  is self-dual.



2.2. We describe a geometric procedure which will be useful for describing the behaviour of Kazhdan-Lusztig elements  $\underline{N}_\mu$  under multiplication by  $C_P$ . Let  $F_0$  be the facet containing  $\mu + \rho$ , and let  $F$  be a facet in the boundary of  $F_0$ . Let  $\Lambda$  be a vector such that the line segment between  $\mu + \rho$  and  $\mu + \rho + \Lambda$  is contained in  $F_0 \cup F \cup F_1$ , with  $F_1$  the facet containing  $\mu + \rho + \Lambda$ . In applications we will also encounter the degenerate cases with  $\mu + \rho \in F$  (i.e. we have a path going from a facet  $F$  of smaller dimension to a facet  $F_1$  which contains  $F$  in its boundary), and the case with  $\mu + \rho + \Lambda \in F$  (i.e. we have a path going from the facet  $F_0$  to a facet in its boundary). Obviously, the general case described first can be obtained as a combination of the two degenerate cases. In the following we will only deal with the general case; the degenerate cases can be treated similarly.

Let  $\mathbf{a}_F(\lambda) = \mathbf{a}_F(F_0)$ , where  $F_0$  is the facet containing  $\lambda + \rho$  (see 1.6), with the definition of  $\mathbf{b}_F(\lambda)$  similar. Then we define

$$(2.4) \quad \mathbf{F}_\Lambda N_\mu = \begin{cases} 0 & \text{if } F \not\subset \mathcal{C}, \\ \sum_{\gamma: \gamma + \rho \in (\lambda + \rho + \Lambda)P} v^{\mathbf{a}_F(\gamma) - \mathbf{b}_F(\lambda)} N_\gamma & \text{if } F \subset \mathcal{C}. \end{cases}$$

We extend the definition of  $\mathbf{F}_\Lambda$  to the elements  $\underline{N}_\mu$  in the following nontrivial way: Observe that  $\underline{N}_\mu$  is a linear combination of elements  $N_\lambda$  with  $\lambda \in \mathcal{W}.\mu$ . By assumption on  $\Lambda$ , the line segment between  $\mu + \rho$  and  $\mu + \rho + \Lambda$  is in the affine subspace spanned by  $F_0$ . Hence if  $w.\mu = \lambda$ , we obtain a well-defined line segment from  $w(\mu + \rho)$  to  $w(\mu + \rho + \Lambda)$ , independent of the choice of  $w$ . We denote the vector given by it by  $\bar{w}(\Lambda)$ , where  $\bar{w}$  is the image of  $w$  in the quotient  $\mathcal{W}/T \cong W$ , with  $T$  being the normal subgroup of translations in  $\mathcal{W}$ . If  $\underline{N}_\mu = \sum_{\lambda \in \mathcal{W}.\mu} n_{\lambda\mu} N_\lambda$ , we define

$$(2.5) \quad \mathbf{F}_\Lambda \underline{N}_\mu = \sum_{\lambda = w.\mu} n_{\lambda\mu} \mathbf{F}_{\bar{w}(\Lambda)} N_\lambda.$$

**Lemma 2.1.** *The element  $\mathbf{F}_\Lambda \underline{N}_\mu = \sum_\gamma u_\gamma N_\gamma$  is self-dual. Moreover, if  $Q$  is the stabilizer of the facet  $F(\mu + \rho)$  containing  $\mu + \rho$ , then  $u_\gamma$  equals the coefficient of  $N_{F(\gamma + \rho)}$  in the expansion of  $[Q]_v^{-1} \underline{N}_{F(\mu + \rho)} C_P$  as in Lemma 1.4(d).*

*Proof.* Let  $\lambda = w.\mu$ , and let  $F'_0, F'$  and  $F'_1$  be the facets obtained by applying  $w$  to  $F_0, F$  and  $F_1$ . Observe that  $\lambda + \rho \in F'_0$  and that the line segment from  $\lambda + \rho$  to  $\lambda + \rho + \bar{w}(\Lambda)$  is contained in  $F'_0 \cup F' \cup F'_1$ . Moreover  $F'_0$  and  $F'$  have right stabilizers  $Q$  and  $P$  respectively. It follows from Lemma 1.4(c) that

$$N_{F'_0} C_P = [Q]_v v^{-\mathbf{b}_{F'}(F'_0)} N_{F'} = [Q]_v \sum_{F'_1} v^{\mathbf{a}_{F'}(F'_1) - \mathbf{b}_{F'}(F'_0)} N_{F'_1},$$

where the summation goes over all facets  $F'_1$  conjugate to  $F_1$  which contain  $F'$  in their boundary. Using the 1-1 correspondence between the facets  $F'_1$  in the sum above, and the points  $\gamma + \rho$  in the orbit of  $\lambda + \rho + \Lambda$  under the

right action of  $P$ , we obtain similarly

$$\mathbf{F}_{\bar{w}(\Lambda)} N_\lambda = \sum_{\gamma} v^{\mathbf{a}_{F'}(\gamma) - \mathbf{b}_{F'}(\lambda)} N_\gamma = \sum_{\gamma + \rho \in F'_1} v^{\mathbf{a}_{F'}(F'_1) - \mathbf{b}_{F'}(F'_0)} N_\gamma.$$

The claim about the coefficient of  $N_{F(\gamma+\rho)} = N_\gamma$  follows from the last two equations, and the definition of  $\mathbf{F}_\Lambda$ , see 2.4 and 2.5. As  $C_P$  is self-dual, so is  $\underline{N}_{F(\mu+\rho)} C_P$ . From this follows that  $\mathbf{F}_\Lambda \underline{N}_\mu$  is self-dual, by the last two equations and the definition of duality on the module  $\mathcal{M}$  (see end of section 2.1).  $\square$

2.3. In order to compute the  $\underline{N}_\mu$  explicitly, we define a ‘positivity’ operator  $\mathbf{P}$  on self-dual linear combinations  $R = \sum_{\lambda \leq \mu} a_\lambda N_\lambda$ , with  $a_\lambda \in \mathbb{Z}[v, v^{-1}]$  and  $a_\mu = 1$  as follows: Let  $i_o$  be the smallest exponent of  $v$  occurring in any of the coefficients  $a_\lambda$  with  $\lambda < \mu$ . If  $i_o > 0$  we define  $\mathbf{P}R = R$ . If  $i_o \leq 0$ , we get rid of the lowest exponent by the operation

$$R \mapsto R - \sum_{\lambda < \mu} (v^{i_o} + v^{-i_o}) [(v^{-i_o} a_\lambda)(0)] \underline{N}_\lambda;$$

if  $i_o = 0$  we subtract from  $R$  the expression  $\sum_{\lambda < \mu} a_\lambda(0) \underline{N}_\lambda$ . Iterating this operation, as long as the lowest exponent is nonpositive, we finally obtain a linear combination  $\mathbf{P}R = \sum_\lambda b_\lambda N_\lambda$  with  $b_\lambda \in v\mathbb{Z}[v]$  for all  $\lambda < \mu$ , i.e. we obtain (cf. Theorem 1.1)

$$(2.6) \quad \mathbf{P}(R) = \underline{N}_\mu.$$

**Theorem 2.2.**  $\underline{N}_{\mu+\Lambda} = \mathbf{P}\mathbf{F}_\Lambda \underline{N}_\mu$  if  $\Lambda$  is in  $\mathcal{C}$  satisfying the conditions at the beginning of Section 2.2.

*Proof.* It was shown in Lemma 2.1 that  $\mathbf{F}_\Lambda \underline{N}_\mu$  is a  $\mathbb{Z}[v, v^{-1}]_0$ -linear combination of  $\underline{N}_\gamma$ s. Moreover, using the fact that  $\Lambda$  is in  $\mathcal{C}$ , we check easily that  $\underline{N}_{\mu+\Lambda}$  has coefficient 1 in  $\mathbf{F}_\Lambda \underline{N}_\mu$ . The claim follows from this and 2.6.  $\square$

### 3. A FAST ALGORITHM

3.1. We now outline how our path version of the Kazhdan-Lusztig algorithm can be used to compute the coefficients  $n_{\lambda\mu}$  in an efficient way. Here, the strategy is to stay at points with stabilizer as large as possible as long as possible.

Let  $\Lambda$  be a vector in  $\mathcal{C}$ . Then we define the operator  $\mathbf{T}_\Lambda : \mathcal{M} \rightarrow \mathcal{M}$  by  $\mathbf{T}_\Lambda \underline{N}_\mu = \underline{N}_{\mu+\Lambda}$ . If for given  $\mu$  the vector  $\Lambda$  is small enough that the line segment between  $\mu + \rho$  and  $\mu + \Lambda + \rho$  satisfies the conditions in Section 2.2,  $\mathbf{T}_\Lambda \underline{N}_\mu = \mathbf{P}\mathbf{F}_\Lambda \underline{N}_\mu$ , by Theorem 2.2. For the general case, we choose a piecewise linear path with vertices  $\mu_0 + \rho = \mu + \rho, \mu_1 + \rho, \dots, \mu_N + \rho = \mu + \Lambda + \rho$  such that each directed line segment from  $\mu_i + \rho$  to  $\mu_{i+1} + \rho$  lies in  $\mathcal{C}$  and satisfies the conditions at the beginning of section 2.2. This can always be achieved if  $\Lambda \in \mathcal{C}$ ; in this case we can write  $\Lambda$  as a linear combination of the fundamental weights with non-negative scalars. So it suffices to chose as

line segments small enough multiples of the fundamental weights. Theorem 2.2 then implies

**Corollary 3.1.**  $\mathbf{T}_\Lambda \underline{N}_\mu = (\mathbf{PF}_{\mu_N - \mu_{N-1}})(\mathbf{PF}_{\mu_{N-1} - \mu_{N-2}}) \cdots (\mathbf{PF}_{\mu_1 - \mu_0}) \underline{N}_\mu$ , independent of the choice of the piecewise linear path.

3.2. A dominant weight  $\mu$  is called *critical* if  $(\mu + \rho) \cdot \alpha$  is divisible by  $l$  for all roots  $\alpha$  of  $\mathfrak{g}$ . In the simply-laced case, the smallest critical weight is the Steinberg weight  $(l-1)\rho$ ; in the nonsimply-laced case the smallest critical weight is a multiple of  $\rho$  or  $\check{\rho}$  (half the sum of the co-roots) depending on whether  $l$  is divisible by the ratio of the square lengths of a long and a short root. In the following, we shall assume  $\mathfrak{g}$  to be *simply-laced* in order to avoid having to deal with various cases. We call a dominant integral weight  $\mu$  *interior* if  $\mu \cdot \alpha_i \geq l$  for all simple roots  $\alpha_i$ . The *fundamental box*  $\Pi$  is the set

$$\left\{ \sum_{i=1}^{k-1} t_i \Lambda_i : 0 < t_i \leq l \right\}.$$

The dominant Weyl chamber is tiled by translates of the fundamental box  $\Pi$  of the form  $\Pi + \mu_c + \rho$ , where  $\mu_c$  is a critical weight in  $\bar{\mathcal{C}}$ . In particular, for each interior weight  $\mu$ , there is a unique critical weight  $\mu_c \in \mathcal{C}$  such that  $\mu + \rho \in \Pi + \mu_c + \rho$ , or, equivalently, with  $\mu \in \mu_c + \Pi$ .

*Case 1: Interior weights* Assume that the critical point  $\mu_c$  of  $\mu + \rho$  is in the interior. Then it is known that  $\underline{N}_{\mu_c} = N_{\mu_c}$  (see [S1], [D1]). We can compute  $\underline{N}_\mu = \mathbf{T}_{\mu - \mu_c} N_{\mu_c}$ , with  $\mathbf{T}_{\mu - \mu_c}$  realized by a suitable piecewise linear path (see subsection 3.1).

*Case 2: Noninterior weights* If  $\mu$  is a non-interior weight, then  $\mu + \rho = \sum_i a_i \Lambda_i$ , with the  $a_i$  non-negative integers with at least one of them  $< l$ . We define scalars  $b_i$  and  $c_i$  as follows: If  $a_i < l$ , then  $b_i = c_i = a_i$ ; otherwise we set  $c_i = l[a_i/l]$  and  $b_i = l$ , where for the purpose of this definition  $[x]$  means the integer part of the real number  $x$ . We then define dominant weights  $\mu_0$  and  $\mu_1$  by

$$\mu_0 + \rho = \sum_i b_i \Lambda_i \quad \text{and} \quad \mu_1 + \rho = \sum_i c_i \Lambda_i;$$

observe that  $\mu_0$  is in the fundamental box, while  $\mu_1$  is close to  $\mu$ , playing the role of the critical weight  $\mu_c$  in case 1. The computation of  $\underline{N}_\mu$  now goes in three steps:

- (i) Compute  $\underline{N}_{\mu_0}$  directly via a suitable piecewise linear path from  $\rho$  to  $\mu_0 + \rho$ .
- (ii) Compute  $\underline{N}_{\mu_1}$  via a suitable piecewise linear path from  $\mu_0 + \rho$  to  $\mu_1 + \rho$ ; suitable here means that the path only goes through facets whose dimensions are at most one higher than the dimension of the facet containing  $\mu_1 + \rho$ .
- (iii) Compute  $\underline{N}_\mu$  via a path from  $\mu_1 + \rho$  to  $\mu + \rho$ .

3.3. The main savings of this algorithm over the usual algorithms comes from step (ii) in the case of noninterior weights, for two reasons: Firstly,

working with facets requires much less memory than having to work with all the alcoves surrounding it separately, and hence also less operations for computing a new Kazhdan-Lusztig element. Secondly, going through a facet of large codimension requires less wall-crossings than having to go around via simple wall-crossings. Indeed, computer experiments for type  $A$  show that this algorithm produces enormous improvements in efficiency over the original LLT algorithm, or the Soergel algorithm (see [GW2] for details).

**3.4.** Let  $\lambda$  be a Young diagram with  $< k$  rows. We identify it with a weight of  $sl_k$  as usual. Taking for  $\Lambda$  a weight of the generating  $k$ -dimensional representation  $V$  of  $sl_k$ , it is not hard to check that the action of the operator  $\mathbf{F}_\Lambda$  on  $\underline{N}_\lambda$  corresponds to the action of one of the operators  $f_i$  on  $\lambda$ , where the  $f_i$ 's describe the action of a Borel algebra of  $U_q \widehat{sl_l}$  on the Fock space consisting of all Young diagrams, see [MM]. Indeed, this action served as motivation for work by [LLT], [Ar] and also for this paper. It would be interesting to find out whether one can find similar interpretations for suitably chosen path operators, as defined in this paper, for other weight lattices.

**3.5. Littelmann path algorithm.** The path operators in this paper can be used to describe the decomposition of the tensor product of an indecomposable tilting module  $T_\mu$  with highest weight  $\mu$  (see [An]) with the fundamental representation  $V$  of  $U_q sl_k$  (as in the previous subsection). Indeed, for given dominant integral weight  $\mu$ , one can find a subset  $\{\varepsilon_i\}_i$  of the weights of  $V$  such that the application of the operator  $\sum_i \mathbf{F}_{\varepsilon_i}$  results into adding *all* possible weights of  $V$  to  $\mu + \rho$ ; e.g. if  $\mu + \rho$  does not lie on any hyperplane, one would have to take as subset *all* weights of  $V$ . Writing  $(\sum_i \mathbf{F}_{\varepsilon_i}) \underline{N}_\mu = \sum_\nu m_\nu \underline{N}_\nu$ , with  $m_\nu \in \mathbb{Z}[v, v^{-1}]_0$  (which is always possible), it follows that the multiplicity of the indecomposable tilting module  $T_\nu$  with highest weight  $\nu$  in  $T_\mu \otimes V$  is equal to  $m_\nu(1)$ . This is an easy consequence of the Littelmann algorithm (see [Li]) in the generic case, and the fact that the multiplicity of an indecomposable tilting module in the tilting module  $T$  is completely determined by the character of  $T$ . Hence, in this case, our algorithm can be seen as an analog of the Littelmann path algorithm for tilting modules (in a rather trivial special case). The same reasoning will also work for other Lie types if one takes for  $V$  a minuscule representation. It would be interesting to see whether one could extend this observation to a description via paths of tensoring  $T_\lambda$  with an arbitrary tilting module  $T$ . Note, however, that even if  $T$  is a simple tilting module, one can not adapt Littelmann's algorithm in a straightforward way in general.

Finally we would like to recall that one obtains the dimension of the simple module  $D^\mu$  of a Hecke algebra of type  $A$  (see [DJ]) as the multiplicity of the tilting module  $T_\mu$  in  $V^{\otimes n}$ . This is a direct consequence of the  $q$ -analogue of Schur duality between Hecke algebras of type  $A$  and  $U_q sl_k$  (see [Do], [DPS]). So the procedure described in the previous paragraph is a method to compute such dimensions inductively. Inductive procedures of this type have already been obtained before by Kleshchev for symmetric

groups in positive characteristic and by Brundan ([Br]) for Hecke algebras by different methods.

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